Exercises for Chapter 6 of *An Introduction to Description Logic*

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Exercise 1 Consider the TBoxes $\mathcal{T}_1 = \mathcal{T}$ and $\mathcal{T}_2 = \mathcal{T} \cup \{A \sqsubseteq C, D \sqsubseteq B\}$ as introduced in the statement of Lemma 6.1.

Show that $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$. Would this still be the case if we added the GCI $A \sqsubseteq B$ to $\mathcal{T}_2$? What about adding $B \sqsubseteq A$?

Exercise 2 Consider the normalisation rules for $\mathcal{EL}$ of Fig. 6.1 and replace rule $\mathsf{NF4}$ by the rule

$$\mathsf{NF4}' \quad C \sqsubseteq D \sqcap E \quad \longrightarrow \quad C \sqsubseteq D, \quad C \sqsubseteq E,$$

i.e., in contrast to rule $\mathsf{NF4}$ the new rule $\mathsf{NF4}'$ does not require the left-hand side of the GCI to which it is applied to be a concept name.

Would Lemma 6.2 still hold if we used $\mathsf{NF4}'$ in place of $\mathsf{NF4}$? What about Proposition 6.5?

Exercise 3 Consider the $\mathcal{EL}$ TBox $\mathcal{T}$ consisting of the following GCIs, where A, B, C, D are concept names:

$$\begin{aligned} A &\sqsubseteq B \sqcap \exists r.C \\ B \sqcap \exists r.B &\sqsubseteq C \sqcap D \\ C &\sqsubseteq (\exists r.A) \sqcap B \\ \exists r.\exists r.B \sqcap D &\sqsubseteq \exists r.(A \sqcap B) \end{aligned}$$

Check whether the following subsumption relationships hold w.r.t. $\mathcal{T}$ by using the subsumption algorithm for $\mathcal{EL}$ introduced in Chapter 6:

1. $A \sqsubseteq B$
2. $A \sqsubseteq \exists r.\exists r.A$
3. $B \sqcap \exists r.A \sqsubseteq \exists r.C$

Exercise 4 Show in detail that the "if" direction of (6.1) in Chapter 6 (page 148) is indeed a consequence of Lemma 6.10.

Exercise 5 Let $\mathcal{T}$ be a general $\mathcal{EL}$ TBox in normal form, $\mathcal{T}^*$ the saturated TBox obtained by exhaustive application of the inference rules of Figure 6.2, $\mathcal{I}_{\mathcal{T}^*}$ the canonical interpretation induced by $\mathcal{T}^*$, and $\mathcal{I}$ an arbitrary model of $\mathcal{T}$. Recall the definition of a simulation introduced in Exercise 8 for Chapter 3, and assume that all concept names in $\mathbf{C}$ and all role names in $\mathbf{R}$ occur in $\mathcal{T}$.

Show that the following is a simulation between $\mathcal{I}_{\mathcal{T}^*}$ and $\mathcal{I}$:

$$A \, \sigma \, d \;\text{ iff }\; d \in A^{\mathcal{I}} \;\text{ for all } A \in \mathbf{C} \cup \{\top\} \text{ and } d \in \Delta^{\mathcal{I}}.$$

Exercise 6 The purpose of this exercise is to show that, in $\mathcal{EL}$, the instance problem can be reduced in polynomial time to the subsumption problem.

Let $\mathcal{T}$ be a general $\mathcal{EL}$ TBox and $\mathcal{A}$ an $\mathcal{EL}$ ABox. For every individual name a occurring in $\mathcal{A}$, we introduce a new concept name N_a, and define:

$$\mathcal{T}_{\mathcal{A}} = \mathcal{T} \cup \{N_a \sqsubseteq C \mid a : C \in \mathcal{A}\} \cup \{N_a \sqsubseteq \exists r.N_b \mid r(a,b) \in \mathcal{A}\}.$$

Show that the following equivalence holds:

$$(\mathcal{T}, \mathcal{A}) \models b : D \;\text{ iff }\; \mathcal{T}_{\mathcal{A}} \models N_b \sqsubseteq D.$$

Would this also work for $\mathcal{ALC}$ in place of $\mathcal{EL}$?

Hint. The direction from right to left of the equivalence (more precisely, its contrapositive) is easy to show. To prove (the contrapositive of) the other direction, first show that, in the canonical model, the new concepts of the form N_a are interpreted as singleton sets.

Exercise 7 Subsumption in $\mathcal{EL}$ is decidable in polynomial time, while subsumption in $\mathcal{ELI}$ is ExpTime-complete, and thus strictly harder than subsumption in $\mathcal{EL}$. Can this be used to show that $\mathcal{ELI}$ is more expressive than $\mathcal{EL}$?

Exercise 8 Show that the following is true for all $\mathcal{ELI}$ concepts C, D and all interpretations $\mathcal{I}$:

$$\mathcal{I} \text{ satisfies } \exists r^-.C \sqsubseteq D \;\text{ iff }\; \mathcal{I} \text{ satisfies } C \sqsubseteq \forall r.D.$$

Exercise 9 Consider the $\mathcal{ELI}$ TBox $\mathcal{T}$ consisting of the following GCIs, where A, A_1, A_2, B, C, D are concept names:

$$\begin{aligned} A_1 \sqcap A_2 &\sqsubseteq \exists r.B \\ \exists r^-.A_2 &\sqsubseteq C \\ A &\sqsubseteq A_1 \sqcap A_2 \\ \exists r.(B \sqcap C) &\sqsubseteq D \end{aligned}$$

Use the subsumption algorithm for $\mathcal{ELI}$ introduced in Chapter 6 to check whether the following subsumption relationships hold w.r.t. $\mathcal{T}$:

1. $A \sqsubseteq D$
2. $\exists r.A \sqsubseteq \exists r.D$
3. $A \sqsubseteq \exists r.A$