Exercises for Chapter 4 of *An Introduction to Description Logic*

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Exercise 1 Given an $\mathcal{ALC}$ TBox $\mathcal{T}$, we say that a concept name A is undefined in $\mathcal{T}$ if $\mathcal{T}$ does not contain a concept definition axiom with A on its left hand side; i.e., $\mathcal{T}$ does not contain an axiom of the form $A \sqsubseteq C$ or $A \equiv C$. Define a procedure for transforming an acyclic $\mathcal{ALC}$ TBox $\mathcal{T}$ into an equivalent TBox $\mathcal{T}'$ in which the concept descriptions on the right hand side of concept definition axioms include only undefined concepts. Prove that

1. this procedure always terminates, and
2. that $\mathcal{T}$ is equivalent to $\mathcal{T}'$ (i.e., that they have the same models).

Hint. consider the decision procedure for acyclic knowledge base consistency presented in 4.2.2.

Exercise 2 Consider the TBox

$$\mathcal{T} := \big\{\neg(A \sqcup B) \sqsubseteq \bot, \quad A \sqsubseteq \neg B \sqcap \exists r.B, \quad D \sqsubseteq \forall r.A, \quad B \sqsubseteq \neg A \sqcap \exists r.A\big\},$$

and the ABox

$$\mathcal{A} := \big\{r(a,b),\ r(a,c),\ r(a,d),\ r(d,c),\ (B \sqcap \forall r.D)(a),\ E(b),\ (\neg A)(c),\ (\exists s.\neg D)(d)\big\},$$

Show how a tableau algorithm would be used to check the consistency of:

1. $\langle \mathcal{T}, \emptyset \rangle$,
2. $\langle \emptyset, \mathcal{A} \rangle$, and
3. $\langle \mathcal{T}, \mathcal{A} \rangle$.

In case any of them is consistent, specify the witness model.

Exercise 3 Show how the tableau algorithm from 4.2.3 can be used to decide whether the following subsumption holds:

$$\neg\forall r.A \sqcap \forall r.C \sqsubseteq_{\mathcal{T}} \forall r.E$$

where $\mathcal{T} = \{C \equiv (\exists r.\neg B) \sqcap \neg A, \quad D \equiv \exists r.B, \quad E \equiv \neg(\exists r.A) \sqcap \exists r.D\}$.

Exercise 4 Prove local correctness for the lazy expansion rule $\equiv_2$.

Exercise 5 Compare the tableau algorithm (with eager expansion) to the tableau algorithm extended with the $\equiv_1$- and $\equiv_2$-rules (lazy expansion) by applying both methods to check whether A is satisfiable w.r.t. $\mathcal{T}$, where

$$\mathcal{T} \coloneqq \{A \equiv \neg B \sqcap B,\ B \equiv \exists r.\exists s.(C \sqcap D)\}.$$

1. What is the maximal number of complete ABoxes obtained in the set of ABoxes by eager expansion? What is the minimal number for lazy expansion?

2. What is the maximal number of rule applications by eager expansion? What is the minimal number for lazy expansion?

3. What is the maximal number of assertions in a complete ABox obtained by eager expansion? What is the minimal number for lazy expansion?

4. Give $\kappa(\mathcal{M})$ for all sets $\mathcal{M}$ of ABoxes considered in the tableau algorithm with eager expansion.

5. Give $\kappa_{\mathcal{T}}(\mathcal{M})$ for all sets $\mathcal{M}$ of ABoxes considered in the tableau algorithm with lazy expansion.

Exercise 6 The tableau algorithm for checking consistency of $\mathcal{ALC}$-ABoxes w.r.t. general TBoxes can be extended to inverse roles by adapting the $\exists$-rule and $\forall$-rule as follows: Let C be an $\mathcal{ALCI}$-concept, and r an $\mathcal{ALCI}$-role, i.e. r denotes a role or an inverse role name, and $(r^{-1})^{-1} = r$ holds.

$\exists$-rule: Condition: $\mathcal{A}$ contains $(\exists r.C)(a)$, a is not blocked, but there is no b with either $\{r(a,b), C(b)\} \subseteq \mathcal{A}$ or $\{r^{-1}(b,a), C(b)\} \subseteq \mathcal{A}$

Action: $\mathcal{A}' \coloneqq \mathcal{A} \cup \{r(a,b), C(b)\}$ for a new individual b not occuring in $\mathcal{A}$

$\forall$-rule: Condition: $(\forall r.C)(a) \in \mathcal{A}$ and $r(a,b) \in \mathcal{A}$ or $r^{-1}(b,a) \in \mathcal{A}$, but $C(b) \notin \mathcal{A}$

Action: $\mathcal{A}' \coloneqq \mathcal{A} \cup \{C(b)\}$

1. Which blocking condition needs to be introduced to obtain a correct decision procedure?

2. Is the extended tableau algorithm for $\mathcal{ALCI}$ sound and complete?

Exercise 7 We consider another form of blocking, where an individual can be blocked by an individual that is not necessarily an ancestor: *anywhere blocking*. Instead of the depth of an individual and the ancestor relation, it uses the age of an individual and the relation $<$.

The *age* of an individual x, denoted by $\mathsf{age}(x)$, is defined as 0 for old individuals and n for a new individual x which was generated by the nth application of the $\exists$-rule.

Let $\mathcal{A}$ be an ABox obtained by applying the tableau rules and the $\sqsubseteq$-rule to an initial ABox $\mathcal{A}_0$. A new individual x is *anywhere blocked* by an individual a in $\mathcal{A}$ iff

- $\{C \mid C(x) \in \mathcal{A}\} \subseteq \{D \mid D(a) \in \mathcal{A}\}$, and
- $\mathsf{age}(a) < \mathsf{age}(x)$.

Prove that the tableau algorithm with anywhere blocking is a decision procedure for consistency of $\mathcal{ALC}$-knowledge bases with general TBoxes. **Hint:** For what subset of the complete tableau do we need to construct a model?

Exercise 8 Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{A}_0 \rangle$ be an $\mathcal{ALC}$-knowledge base, where $\mathcal{T}$ is a general TBox. The *precompletion of* $\mathcal{K}$ is the set of ABoxes $\mathcal{M}$ that is produced by the tableau algorithm when starting with the set of ABoxes $\{\mathcal{A}_0\}$ and exhaustively applying all tableau rules except the modified $\exists$-rule.

1. Show that $\mathcal{K}$ is consistent iff there is an open ABox $\mathcal{A} \in \mathcal{M}$ such that for all individual names a occurring in $\mathcal{A}$, the concept $C^a_{\mathcal{A}} := \bigsqcap_{C(a)\in\mathcal{A}} C$ is satisfiable w.r.t. $\mathcal{T}$.

   Hint: For the "if" direction, proceed as follows: The correctness of the tableau algorithm for $\mathcal{ALC}$ implies that, if $C^a_{\mathcal{A}}$ is satisfiable, then exhaustively applying all (!) rules to the set of ABoxes $\{\{C^a_{\mathcal{A}}(a)\}\}$ yields a set $\mathcal{M}'$ that contains an open and complete ABox. Show how to join all these ABoxes to obtain an open and complete tableau for $\mathcal{A}$ and conclude that $\mathcal{A}_0$ is consistent w.r.t. $\mathcal{T}$.

2. Use the result from a) to prove that ABox consistency in $\mathcal{ALC}$ can be decided in deterministic exponential time.

Exercise 9 Show that the size of $|C|_{\mathcal{T}}$ of a concept C w.r.t. an acyclic TBox $\mathcal{T}$ is well-defined.

Exercise 10 For each of the following $\mathcal{ALC}$-concept descriptions C and $\mathcal{ALC}$-TBoxes $\mathcal{T}$, decide whether C is satisfiable w.r.t. $\mathcal{T}$ by using the tableau algorithm for ABox consistency (Section 4.2.1), acyclic KB consistency (Section 4.2.2) or general KB consistency (Section 4.2.3) in order to construct a model or show that no model exists. In each case use the simplest applicable algorithm, and explain why it is applicable.

1. $C := A$
   $\mathcal{T} := \{A \sqsubseteq \neg A\}$
2. $C := A$
   $\mathcal{T} := \emptyset$

3. $C \coloneqq A \sqcap \exists r.A$
   $\mathcal{T} \coloneqq \{A \sqsubseteq \forall r.\neg A\}$

4. $C \coloneqq A \sqcap \exists r.(B \sqcup \exists r.C)$
   $\mathcal{T} \coloneqq \{A \sqsubseteq \forall r.\neg B\}$

5. $C \coloneqq A \sqcap \exists r.(B \sqcup \exists r.C)$
   $\mathcal{T} \coloneqq \{A \sqsubseteq \forall r.\neg B, \neg B \sqsubseteq \forall r.\neg C\}$

Exercise 11 Use a tableau algorithm to decide whether the following $\mathcal{ALC}$-knowledge base is consistent:

$$\mathcal{T} \coloneqq \{A \sqcap \forall r.\neg A \sqsubseteq \bot\}$$
$$\mathcal{A} \coloneqq \{(\forall r.\neg A)(a),\ (\exists r.A)(b),\ r(a,b)\}$$