Exercises for Chapter 2 of *An Introduction to Description Logic*

June 5, 2017

1 Exercises for Section 2.1

Exercise 1 The goal of this exercise is to make sure that you understand the notion of an *interpretation*.

1. How many elements does the smallest domain of an interpretation contain?
2. Can an interpretation domain be infinite?
3. In an interpretation $\mathcal{I}$ and for a concept name A, how many elements can/must $A^{\mathcal{I}}$ have?
4. In an interpretation $\mathcal{I}$ and for a role name r, how many pairs of elements can/must $r^{\mathcal{I}}$ have?
5. For an element $e \in \Delta^{\mathcal{I}}$, can it be the case that $(e, e) \in r^{\mathcal{I}}$?
6. For two elements $e, f \in \Delta^{\mathcal{I}}$, can it be the case that $\{(e, f), (f, e)\} \subseteq r^{\mathcal{I}}$?

Solution

1. One element
2. Yes
3. Any number between 0 and the number of elements in $\Delta^{\mathcal{I}}$
4. Any number between 0 and the number of elements in $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
5. Yes
6. Yes

Exercise 2 Formulate $\mathcal{ALC}$ concepts: for each of the following concepts, build a suitable $\mathcal{ALC}$ concept description, using only the concept names

$$\mathsf{Person}, \mathsf{Happy}, \mathsf{Animal}, \mathsf{Cat}, \mathsf{Old}, \mathsf{Fish}$$

and the role name *owns*.

1. happy person
2. happy pet owner
3. person who owns only cats
4. unhappy pet owners who own an old cat
5. pet owners who only own cats and fish

Solution

1. $\mathsf{Person} \sqcap \mathsf{Happy}$
2. $\mathsf{Person} \sqcap \mathsf{Happy} \sqcap \exists \mathit{owns}.\mathsf{Animal}$
3. $\mathsf{Person} \sqcap \forall \mathit{owns}.\mathsf{Cat}$
4. $\mathsf{Person} \sqcap (\neg\mathsf{Happy}) \sqcap \exists \mathit{owns}.(\mathsf{Animal} \sqcap \mathsf{Cat} \sqcap \mathsf{Old})$ (please note that the "$\mathsf{Animal}\sqcap$" is not strictly necessary but included here to ensure that this concept indeed describes a pet owner)
5. $\mathsf{Person} \sqcap \exists \mathit{owns}.\mathsf{Animal} \sqcap \forall \mathit{owns}.(\mathsf{Cat} \sqcup \mathsf{Fish})$ (please note how the English "and" is often read as a logical "or", like in this case here)

Exercise 3 For each of the concepts formulated as answers of Exercise 2, draw an interpretation that has an element in the extension of that concept.

Solution interpretations.

1. In all three interpretations given in Figure 1, m is the only element in the extension of $\mathsf{Person} \sqcap \mathsf{Happy}$.
2. In all three interpretations given in Figure 1, m is the only element in the extension of $\mathsf{Person} \sqcap \mathsf{Happy} \sqcap \exists \mathit{owns}.\mathsf{Animal}$.
3. In all three interpretations given in Figure 1, n and s are elements in the extension of $\mathsf{Person} \sqcap \forall \mathit{owns}.\mathsf{Cat}$ (since they own nothing), and m and r are elements in the extension of $\mathsf{Person} \sqcap \forall \mathit{owns}.\mathsf{Cat}$ in the bottom two interpretations.
4. Only the middle interpretation given in Figure 1 has an instance of $\mathsf{Person} \sqcap (\neg\mathsf{Happy}) \sqcap \exists \mathit{owns}.(\mathsf{Animal} \sqcap \mathsf{Cat} \sqcap \mathsf{Old})$, namely m.
5. Only the last interpretation given in Figure 1 has instances of $\mathsf{Person} \sqcap \exists \mathit{owns}.\mathsf{Animal} \sqcap \forall \mathit{owns}.(\mathsf{Cat} \sqcup \mathsf{Fish})$, namely m and r.

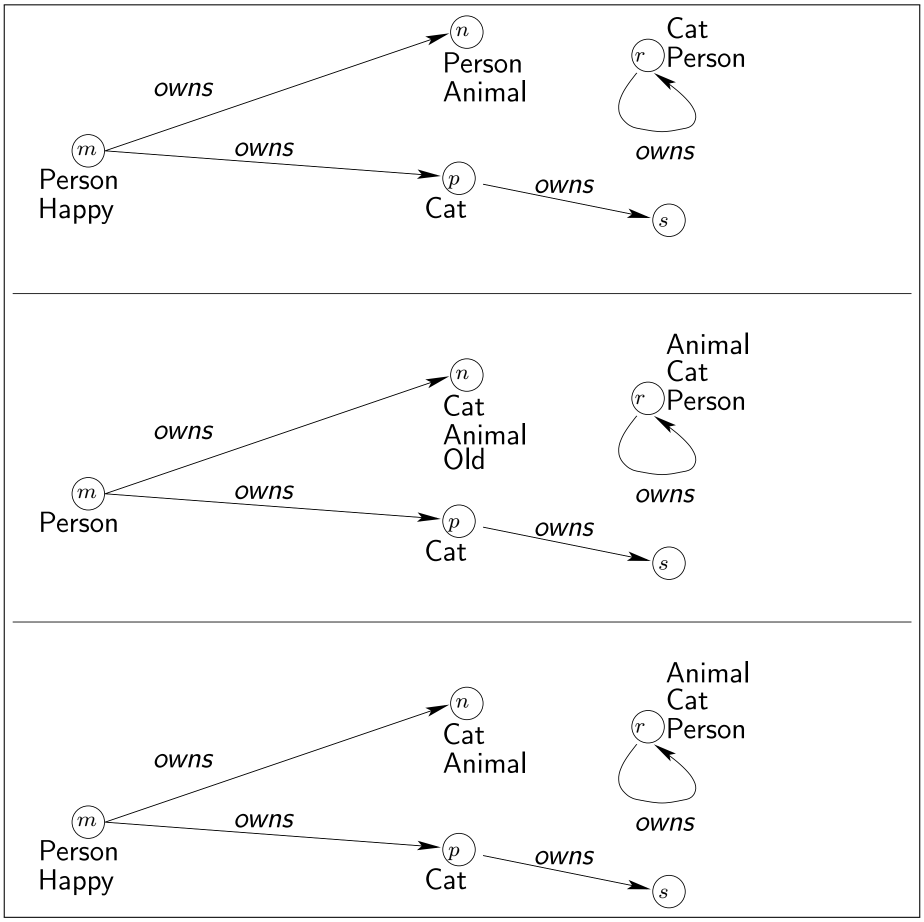

Figure 1: Two interpretations as solutions to Exercise 3.

2 Exercises for Section 2.2

Exercise 4 Build an $\mathcal{ALC}$ knowledge base: capture each of the following statements in a suitable GCI, equivalence axioms, or assertion, using only the concept names

Vehicle, Boat, Bicycle, Car, Device, Wheel, Engine, Axle , Rotation, Water
Human, Driver, Adult, Child

and the role names

hasPart, *poweredBy*, *capableOf* , *travelsOn controls*.

1. Cars are exactly those vehicles that have wheels and are powered by an engine.

2. Bicycles are exactly those vehicles that have wheels and are powered by a human.
3. Boats are exactly those vehicles that travel on water.
4. Boats have no wheels.
5. Cars and bicycles do not travel on water.
6. Wheels are exactly those devices that have an axle and are capable of rotation.
7. Drivers are exactly those humans who control a vehicle.
8. Drivers of cars are adults.
9. Humans are not vehicles.
10. Wheels or engines are not humans.
11. Humans are either adults or children.
12. Adults are not children.
13. Bob controls a car.
14. Bob is a human.
15. Bob controls QE2.
16. QE2 is a vehicle that travels on water.

Solution of 'and' and 'or', especially in (5) and (10). Also, please note that we have normalised our concept names to be singular and took care to pick suitable role names.

1. $\mathsf{Car} \equiv \mathsf{Vehicle} \sqcap \exists \mathit{hasPart}.\mathsf{Wheel} \sqcap \exists \mathit{poweredBy}.\mathsf{Engine}$
2. $\mathsf{Bicycle} \equiv \mathsf{Vehicle} \sqcap \exists \mathit{hasPart}.\mathsf{Wheel} \sqcap \exists \mathit{poweredBy}.\mathsf{Human}$
3. $\mathsf{Boat} \equiv \mathsf{Vehicle} \sqcap \exists \mathit{travelsOn}.\mathsf{Water}$
4. $\mathsf{Boat} \sqsubseteq \forall \mathit{hasPart}.\neg\mathsf{Wheel}$
5. $\mathsf{Car} \sqcup \mathsf{Bicycle} \sqsubseteq \forall \mathit{travelsOn}.\neg\mathsf{Water}$
6. $\mathsf{Wheel} \equiv \mathsf{Device} \sqcap \exists \mathit{hasPart}.\mathsf{Axle} \sqcap \exists \mathit{capableOf}.\mathsf{Rotation}$
7. $\mathsf{Driver} \equiv \mathsf{Human} \sqcap \exists \mathit{controls}.\mathsf{Vehicle}$
8. $\mathsf{Driver} \sqcap \exists \mathit{controls}.\mathsf{Car} \sqsubseteq \mathsf{Adult}$ (alternatively, $\mathsf{Person} \sqcap \exists \mathit{controls}.\mathsf{Car} \sqsubseteq \mathsf{Adult}$)
9. $\mathsf{Human} \sqsubseteq \neg\mathsf{Vehicle}$

10. $\mathsf{Wheel} \sqcup \mathsf{Engine} \sqsubseteq \neg\mathsf{Human}$

11. $\mathsf{Human} \sqsubseteq \mathsf{Adult} \sqcup \mathsf{Child}$

12. $\mathsf{Adult} \sqsubseteq \neg\mathsf{Child}$

13. $\mathsf{Bob} : (\exists \mathit{controls}.\mathsf{Car})$

14. $\mathsf{Bob} : \mathsf{Human}$

15. $(\mathsf{Bob}, \mathsf{QE2}) : \mathit{controls}$

16. $\mathsf{QE2} : (\mathsf{Vehicle} \sqcap \exists \mathit{travelsOn}.\mathsf{Water})$

Exercise 5 Which of the statements in your answer to Exercise 4 are GCIs, equivalence axioms, concept assertions, or role assertions? Is the TBox of your knowledge base acyclic? If yes, can you unfold it into the ABox of you knowledge base?

Solution The TBox of our knowledge base are the statements (1) - (12), of which the statements (1) - (3), (6), and (7) are equivalence axioms, and statements (4), (5), (8) - (12) are GCIs. The statements (13) - (16) constitutes its ABox, of which all statements but (15) are concept assertions, and statement (15) is a role assertion.

The TBox of our knowledge base is *not* acyclic: it contains GCIs, and in particular it contains GCIs (5) and (8) with compound concept descriptions on their left hand side, i.e., they cannot be directly transformed into a concept definition using Lemma 2.3. Also, our TBox contains both an equivalence axiom and a GCI with Driver on their left hand side.

Exercise 6 Draw a model of your answer to Exercise 6. Modify it such that it is no longer a model, in three different ways.

Solution In Figure 2, you will find an example interpretations that happens to be a model of our knowledge base given as solution to Exercise 4 . Please note that, in this model, we have more cars, more drivers, and more humans than what is strictly required by our knowledge base. Also, we have some oddities, e.g., we have two cars sharing a wheel, and cars with a single wheel only. However, this interpretation clearly satisfies all statements in our knowledge base and is therefor a model of it.

There are many ways to turn this interpretation into one that is not a model of our knowledge base, e.g., remove m from the extension of Human or of Human or of Driver. Alternatively, remove (m, p) from the extension of $\mathit{controls}$, etc.

3 Exercises for Section 2.3

Exercise 7 Which of the following concepts is satisfiable?

1. $A \sqcap \neg A$

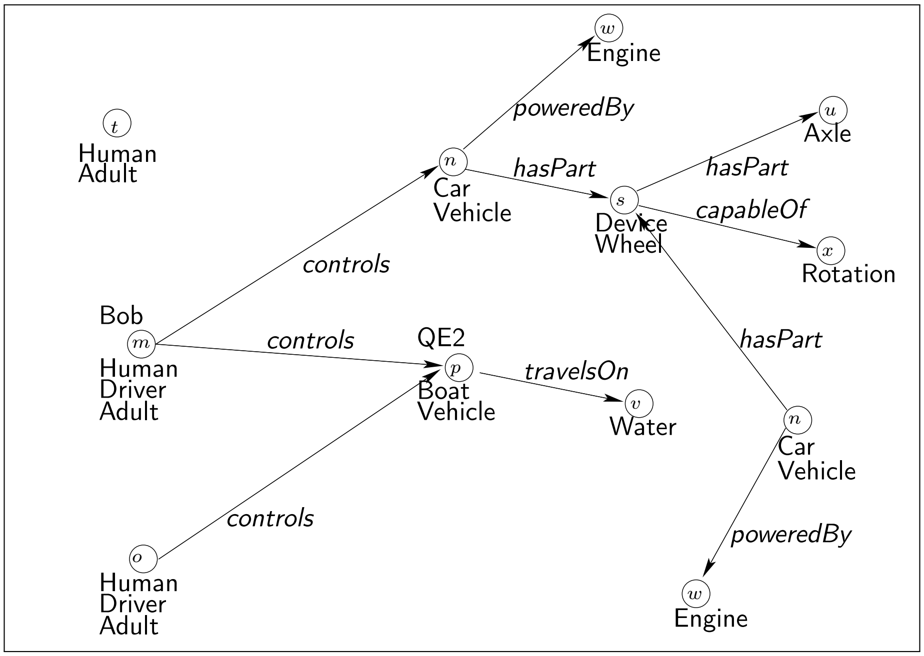

Figure 2: An interpretations as solutions to Exercise 6.

2. $A \sqcup \neg A$
3. $A \sqcap \exists r.B \sqcap \exists r.\neg B$
4. $A \sqcap \exists r.B \sqcap \forall s.\neg B$
5. $A \sqcap \exists r.B \sqcap \forall r.\neg B$
6. $A \sqcap \exists r.B \sqcap \forall r.(\neg B \sqcup \exists r..A)$
7. $A \sqcap \exists r.(B \sqcap C) \sqcap \forall r.\neg B$

Solution

1. $A \sqcap \neg A$ is not satisfiable
2. $A \sqcup \neg A$ is satisfiable
3. $A \sqcap \exists r.B \sqcap \exists r.\neg B$ is satisfiable
4. $A \sqcap \exists r.B \sqcap \forall s.\neg B$ is satisfiable
5. $A \sqcap \exists r.B \sqcap \forall r.\neg B$ is not satisfiable
6. $A \sqcap \exists r.B \sqcap \forall r.(\neg B \sqcup \exists r.A)$ is satisfiable

7. $A \sqcap \exists r.(B \sqcap C) \sqcap \forall r.\neg B$ is not satisfiable

Exercise 8 Which of the following statements is true?

1. $A \sqcap \neg A$ is subsumed by B
2. B is subsumed by $A \sqcup \neg A$
3. $A \sqcap \exists r.B$ is subsumed by $A \sqcap \exists r.\top$
4. $A \sqcap \exists r.(B \sqcap C)$ is subsumed by $A \sqcap \exists r.B$
5. $A \sqcap \exists r.(B \sqcup C)$ is subsumed by $A \sqcap \exists r.B$
6. $A \sqcap \forall r.B$ is subsumed by $A \sqcap \exists r.B$
7. $A \sqcap \exists r.B$ is subsumed by $A \sqcap \forall r.B$
8. $A \sqcap \exists r.A \sqcap \forall r.B$ is subsumed by $A \sqcap \exists r.B$

Solution

1. true
2. true
3. true
4. true
5. false
6. false
7. false
8. true

Exercise 9 Consider again the knowledge base $\mathcal{K}$ given as solution to Exercise 4. Which of the following statements is true?

1. $\mathcal{K}$ is consistent.
2. the concept $\mathsf{Boat} \sqcap \exists \mathit{hasPart}.\mathsf{Wheel}$ is satisfiable w.r.t. $\mathcal{K}$.
3. the concept $\mathsf{Boat} \sqcap \exists \mathit{poweredBy}.\mathsf{Engine}$ is satisfiable w.r.t. $\mathcal{K}$.
4. the concept $\mathsf{Car} \sqcap \mathsf{Bicycle}$ is satisfiable w.r.t. $\mathcal{K}$.
5. the concept $\mathsf{Driver} \sqcap \mathsf{Vehicle}$ is satisfiable w.r.t. $\mathcal{K}$.
6. the concept $\mathsf{Driver} \sqcap \mathsf{Child}$ is satisfiable w.r.t. $\mathcal{K}$.
7. the concept $\exists \mathit{controls}.\mathsf{Car} \sqcap \mathsf{Child}$ is satisfiable w.r.t. $\mathcal{K}$.

8. the concept $\exists \mathit{controls}.\mathsf{Car} \sqcap \mathsf{Child} \sqcap \mathsf{Human}$ is satisfiable w.r.t. $\mathcal{K}$.

9. Bob is an instance of Adult w.r.t. $\mathcal{K}$.

10. Bob is an instance of Driver w.r.t. $\mathcal{K}$.

11. Bob is an instance of (Adult $\sqcap$ Driver) w.r.t. $\mathcal{K}$.

12. QE2 is an instance of Boat w.r.t. $\mathcal{K}$.

13. Driver is subsumed by Human w.r.t. $\mathcal{K}$.

14. Adult is subsumed by Human w.r.t. $\mathcal{K}$.

15. $\mathsf{Human} \sqcap \exists \mathit{controls}.(\mathsf{Vehicle} \sqcap \exists \mathit{hasPart}.\mathsf{Wheel} \sqcap \exists \mathit{poweredBy}.\mathsf{Engine})$ is subsumed by Adult w.r.t. $\mathcal{K}$ (this is a difficult one!).

16. $\exists \mathit{controls}.\mathsf{Car}$ is subsumed by Adult w.r.t. $\mathcal{K}$ (this is another difficult one!).

Solution

1. true
2. false
3. true
4. true
5. false
6. true
7. true
8. false
9. true
10. true
11. true
12. true
13. true
14. false
15. true
16. false

4 Exercises for Section 2.5

Exercise 10 Extend the knowledge base you built in Exercise 4 to capture the following statements (you may need more than one axiom for some of the statements below).

1. Cars have between three and four wheels.
2. Bicycles have exactly two wheels.
3. A human who legally controls a car holds a driving license and is an adult (this is a difficult one!).
4. A vehicle is controlled by exactly one human.
5. A thing's parts' parts are that thing's parts.
6. A car with a broken part is broken.
7. Bob controls a car with a wheel that has a broken axle.

Solution

1. $\mathsf{Car} \sqsubseteq (\geqslant 3\ \mathit{hasPart}.\mathsf{Wheel}) \sqcap (\leqslant 3\ \mathit{hasPart}.\mathsf{Wheel})$
2. $\mathsf{Bicycle} \sqsubseteq (=2\ \mathit{hasPart}.\mathsf{Wheel})$
3. $\mathit{legallyControls} \sqsubseteq \mathit{controls}$, $\mathsf{Human} \sqcap \exists \mathit{legallyControls}.\mathsf{Car} \sqsubseteq \mathsf{Adult} \sqcap \exists \mathit{owns}.\mathsf{DrivingLicense}$
4. $\mathsf{Vehicle} \sqsubseteq (\leqslant 1\ \mathit{controls}^{-}.\mathsf{Human})$
5. $\mathsf{Trans}(\mathit{hasPart})$
6. $\mathsf{Vehicle} \sqcap \exists \mathit{hasPart}.\mathsf{Broken} \sqsubseteq \mathsf{Broken}$
7. $\mathsf{Bob} : (\exists \mathit{controls}.(\mathsf{Car} \sqcap \exists \mathit{hasPart}.(\mathsf{Wheel} \sqcap \exists \mathit{hasPart}.(\mathsf{Axle} \sqcap \mathsf{Broken}))))$

Exercise 11 Consider the knowledge base $\mathcal{K}'$ that is the result of your answers to Exercise 4 and 10: which of the following statements is true?

1. $\mathcal{K}$ is consistent.
2. $\exists \mathit{legallyControls}.\top$ is subsumed by $\exists \mathit{controls}.\top$ w.r.t. $\mathcal{K}'$.
3. the concept $\mathsf{Car} \sqcap \mathsf{Bicycle}$ is satisfiable w.r.t. $\mathcal{K}'$.
4. Bob is an instance of $\exists \mathit{controls}.(\mathsf{Car} \sqcap \mathsf{Broken})$ w.r.t. $\mathcal{K}'$.
5. the interpretation given in Figure ch2-fig2 is a model of $\mathcal{K}'$.

Solution

1. true

2. true
3. false
4. true
5. false (for example, this interpretation does not satisfies the new GCIs (1) and (4).

5 Exercises for Section 2.6

Exercise 12 Translate the knowledge base given as answer to Exercise 4 into Modal Logic.

Solution

$$
\begin{aligned}
[U](\mathsf{Car} &\Leftrightarrow \mathsf{Vehicle} \wedge \langle \mathit{hasPart} \rangle \mathsf{Wheel} \wedge \langle \mathit{poweredBy} \rangle \mathsf{Engine}) \wedge \\
[U](\mathsf{Bicycle} &\Leftrightarrow \mathsf{Vehicle} \wedge \langle \mathit{hasPart} \rangle \mathsf{Wheel} \wedge \langle \mathit{poweredBy} \rangle \mathsf{Human}) \wedge \\
[U](\mathsf{Boat} &\Leftrightarrow \mathsf{Vehicle} \wedge \langle \mathit{travelsOn} \rangle \mathsf{Water}) \wedge \\
[U](\mathsf{Boat} &\Rightarrow [\mathit{hasPart}] \neg \mathsf{Wheel}) \wedge \\
[U](\mathsf{Car} \vee \mathsf{Bicycle} &\Rightarrow [\mathit{travelsOn}] \neg \mathsf{Water}) \wedge \\
[U](\mathsf{Wheel} &\Leftrightarrow \mathsf{Device} \wedge \langle \mathit{hasPart} \rangle \mathsf{Axle} \wedge \langle \mathit{capableOf} \rangle \mathsf{Rotation}) \wedge \\
[U](\mathsf{Driver} &\Leftrightarrow \mathsf{Human} \wedge \langle \mathit{controls} \rangle \mathsf{Vehicle}) \wedge \\
[U](\mathsf{Driver} \wedge \langle \mathit{controls} \rangle \mathsf{Car} &\Rightarrow \mathsf{Adult}) \wedge \\
[U](\mathsf{Human} &\Rightarrow \neg \mathsf{Vehicle}) \wedge \\
[U](\mathsf{Wheel} \vee \mathsf{Engine} &\Rightarrow \neg \mathsf{Human}) \wedge \\
[U](\mathsf{Human} &\Rightarrow (\mathsf{Adult} \vee \mathsf{Child})) \wedge \\
[U](\mathsf{Adult} &\Rightarrow \neg \mathsf{Child}) \wedge \\
&@_{\mathsf{Bob}}(\langle \mathit{controls} \rangle \mathsf{Car}) \wedge \\
&@_{\mathsf{Bob}} \mathsf{Human} \wedge \\
&@_{\mathsf{Bob}}(\langle \mathit{controls} \rangle \mathsf{QE2}) \wedge \\
&@_{\mathsf{QE2}}(\mathsf{Vehicle} \wedge \langle \mathit{travelsOn} \rangle \mathsf{Water})
\end{aligned}
$$

Exercise 13 Translate the knowledge base given as answer to Exercise 4 into First Order Logic.

Solution

$$
\begin{aligned}
\forall x.(\mathsf{Car}(x) &\Leftrightarrow \mathsf{Vehicle}(x) \wedge \exists y.\textsf{\textit{hasPart}}(x,y) \wedge \mathsf{Wheel}(y) \wedge \\
&\quad\ \exists y.\textsf{\textit{poweredBy}}(x,y) \wedge \mathsf{Engine}(y)) \wedge \\
\forall x.(\mathsf{Bicycle}(x) &\Leftrightarrow \mathsf{Vehicle}(x) \wedge \exists y.\textsf{\textit{hasPart}}(x,y) \wedge \mathsf{Wheel}(y) \wedge \\
&\quad\ \exists y.\textsf{\textit{poweredBy}}(x,y) \wedge \mathsf{Human}(y)) \wedge \\
\forall x.(\mathsf{Boat}(x) &\Leftrightarrow \mathsf{Vehicle}(x) \wedge \exists y.\textsf{\textit{travelsOn}}(x,y) \wedge \mathsf{Water}(y)) \wedge \\
\forall x.(\mathsf{Boat}(x) &\Rightarrow \forall y.\textsf{\textit{hasPart}}(x,y) \Rightarrow \neg\mathsf{Wheel}(y)) \wedge \\
\forall x.((\mathsf{Car}(x) \vee \mathsf{Bicycle}(x)) &\Rightarrow \forall y.\textsf{\textit{travelsOn}}(x,y) \Rightarrow \neg\mathsf{Water}(y)) \wedge \\
\forall x.(\mathsf{Wheel}(x) &\Leftrightarrow \mathsf{Device}(x) \wedge \exists y.\textsf{\textit{hasPart}}(x,y) \wedge \mathsf{Axle}(y) \wedge \\
&\quad\ \exists y.\textsf{\textit{capableOf}}(x,y) \wedge \mathsf{Rotation}(y)) \wedge \\
\forall x.(\mathsf{Driver}(x) &\Leftrightarrow \mathsf{Human}(x) \wedge \exists y.\textsf{\textit{controls}}(x,y) \wedge \mathsf{Vehicle}(y)) \wedge \\
\forall x.(\mathsf{Driver}(x) \wedge \exists y.\textsf{\textit{controls}}&(x,y) \wedge \mathsf{Car}(y) \Rightarrow \mathsf{Adult}(x)) \wedge \\
\forall x.(\mathsf{Human}(x) &\Rightarrow \neg\mathsf{Vehicle}(x)) \wedge \\
\forall x.(\mathsf{Wheel}(x) \vee \mathsf{Engine}(x) &\Rightarrow \neg\mathsf{Human}(x)) \wedge \\
\forall x.(\mathsf{Human}(x) &\Rightarrow \mathsf{Adult}(x) \vee \mathsf{Child}(x)) \wedge \\
\forall x.(\mathsf{Adult}(x) &\Rightarrow \neg\mathsf{Child}(x)) \wedge \\
(\exists y.\textsf{\textit{controls}}&(\mathsf{Bob},y) \wedge \mathsf{Car}(y)) \wedge \\
\mathsf{Human}&(\mathsf{Bob}) \wedge \\
\textsf{\textit{controls}}&(\mathsf{Bob},\mathsf{QE2}) \wedge \\
\mathsf{Vehicle}(\mathsf{QE2}) \wedge \exists y.\textsf{\textit{travelsOn}}&(\mathsf{QE2},y) \wedge \mathsf{Water}(y))
\end{aligned}
$$